M.Sc Sem-III. Unit-02
Paper XI.

Continuous or (Bounded) linear transformations

let N and N' be normed linear spaces with the same scalars. A linear transformation T of N into N' is said to be continuous, iff it is continuous as a mapping to the metric space N into the metric space N'. This is equivalent to saying that T is continuous if and only if for any sequence $\langle x_n \rangle$ in N converging to $x \in N$, the sequence $\langle T(x_n) \rangle$ in N' converges to $T(x) \in N'$. If there exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|$ for every $x \in N$, then K is called a bound for T and such a T is often refered to as bounded linear transformation.

1. Theorem:- let T be a linear transformation of a normed linear space N into another normed linear space N'. Then the following statements are equivalent to one another.

(i) T is continuous

(ii) T is continuous at the origin, In the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$

(iii) There exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|$ for all $x \in N$.

that is, T is bounded.

(iv) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image is a bounded set in N'.

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Proof. (i) ⟺ (ii) : Let T be continuous and suppose $\langle x_n \rangle$ is any sequence in N such that $x_n \to 0$ as $n \to \infty$. Then by continuity of T, we have $x \to 0 \Rightarrow T(x_n) \to T(0) = 0$. Hence T is continuous at the origin.

Conversely let T be continuous at the origin and $\langle x_n \rangle$ be any sequence in N such that $x_n \to x \in N$. Then $x_n - x \to 0 \Rightarrow T(x_n - x) = 0$ ($\because$ T is continuous at origin)

$$\Rightarrow T(x_n) - T(x) = 0 \Rightarrow T(x_n) = T(x).$$

Showing that T is a continuous mapping.

(ii) ⟺ (iii) : Let T be continuous at the origin and suppose if possible, T is not bounded, that is, that there exists no real number K such that $\|T(x)\| \leq K\|x\|$ for every $x \in N$. Then for each positive integer n, we can find a vector x_n, such that

$$\|T(x_n)\| > n\|x_n\|$$

$$\Rightarrow \frac{1}{n\|x_n\|} \|T(x_n)\| > 1$$

$$\Rightarrow \left\| \frac{1}{n\|x_n\|} T(x_n) \right\| > 1,$$

$$\left[\text{Note that } \left| \frac{1}{n\|x_n\|} \right| = \frac{1}{n\|x_n\|} \right]$$

$$\Rightarrow \left\| T\left(\frac{x_n}{n\|x_n\|} \right) \right\| > 1.$$

[$\because$ $\alpha T(x) = T(\alpha x)$ for any scalar α)

Now let $y_n = x_n / n\|x_n\|$. Then $\|y_n\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n} \to 0$ as $n \to \infty$.

And so $y_n \to 0$ as $n \to \infty$. But $T(y_n)$ does not tend to 0, since $\|T(y_n)\| > 1$.

Hence T is not continuous at the origin which is a contradiction. Hence T must be bounded.

Conversely, let T be bounded so that there exists a real number $k > 0$ such that

$$\|T(x)\| \leq k\|x\| \;\forall\; x \in N. \quad \text{——— (1)}$$

To show that T is continuous at the origin, let $\langle x_n \rangle$ be any sequence in N such that $x \to 0$. Then

$$\|x_n\| \to \|0\| = 0$$

Also from (1),

$$\|T(x_n)\| \leq K\|x_n\| \;\forall\; n \quad \text{——— (2)}$$

It follows that $\|T(x_n)\| \to 0$ which implies that $T(x_n) \to 0$ we have thus shown that $x_n \to 0$ $\Rightarrow T(x_n) \to 0$ and consequently T is continuous at the origin.

(iii) $\Leftrightarrow$ (iv) : Assume that $\|T(x)\| \leq k\|x\|$ for every $x \in N$ and let x be any point of the closed unit sphere S so that $\|x\| \leq 1$. Then $\|T(x)\| \leq K$ for all $x \in S$. It follows that $T[S]$ is a bounded set in N'.

Conversely, let $T[S]$ be bounded so that there exists a real number $K \geq 0$ such that

$$\|T(x)\| \leq K \text{ for all } x \in S \quad \text{——— (3)}$$

If $x = 0$ then $T(x) = 0$ and so clearly $\|T(x)\| \leq K\|x\|$.

And if $x \neq 0$, then $x/\|x\| \in S$. $[\because \|(x/\|x\|)\| = 1]$

And therefore by (3)

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \le k \Rightarrow \left\|\frac{1}{\|x\|}T(x)\right\| \le K$$

$$\Rightarrow \frac{1}{\|x\|}\|T(x)\| \le K$$

$$\Rightarrow \|T(x)\| \le k\|x\| .$$

Thus it is show that $\|T(x)\| \le K\|x\|$ for all $x \in N$.

Hence T is bounded and the proof is complete.

8. Theorem: Let N and N' be normed linear spaces and let T be a linear transformation of N into N'. Then the inverse T^{-1} exists and is continuous on its domain of definition if and only if there exists a constant $m > 0$ such that $m\|x\| \le \|T(x)\|$ for all $x \in N$. —— (1)

Proof:- Let (1) holds. To show that T^{-1} exists and is continuous. Now T^{-1} exists iff T is one-one.

Let $x_1, x_2 \in N$. Then $T(x_1) + T(x_2) \Rightarrow T(x_1) - T(x_2) = 0$

$$\Rightarrow T(x_1 - x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ by (1)} \Rightarrow x_1 = x_2 .$$

Hence T is one-one and so T^{-1} exists. Therefore to each y in the $T(x) = y \Rightarrow T^{-1}(y) = x$ —— (2)

Hence (1) is equivalent to

$$m\|T^{-1}(y)\| \le \|y\| \Rightarrow \|T^{-1}(y)\| \le \frac{1}{m}\|y\|$$

$\Rightarrow T^{-1}$ is bounded $\Rightarrow T^{-1}$ is continuous.

Conversely, let T^{-1} exist ~~must~~ be and be continuous on its domain $T[N]$. Let x be an arbitrary element in N. Since T^{-1} exists, there is $y \in T[N]$ such that

$$T^{-1}(y) = x \Leftrightarrow T(x) = y$$

Again since T^{-1} is continuous, it is bounded so that there exists a positive constant K such that

$$\|T^{-1}(y)\| \leq K\|y\| \Rightarrow \|x\| \leq K\|T(x)\|$$

$$\Rightarrow m\|x\| \leq \|T(x)\| \text{ where } m = \frac{1}{K} > 0.$$

Proved.

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